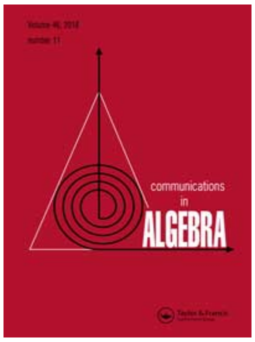

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Lourdes Juan[a] and Andy Magid[b]

[a]Department of Mathematics, Texas Tech University, Lubbock, Texas, USA; [b]Department of Mathematics, University of Oklahoma, Norman, Oklahoma, USA

ABSTRACT
Differential modules over a commutative differential ring which are projective as ring modules, with differential homomorphisms, form an additive category. Every projective ring module is shown occurs as the underlying module of a differential module. Differential modules, projective as ring modules, are shown to be direct summands of differential modules free as ring modules; those which are differential direct summands of differential direct sums of the ring being induced from the subring of constants. Every differential module finitely generated and projective as a ring module is shown to have this form after a faithfully flat finitely presented differential extension of the base.

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1. Introduction

A differential module over a differential ring is a module equipped with an (abelian group) endomorphism which satisfies a product rule (with respect to the ring derivation) for scalar multiplication. These have been studied in many places: for example [10], [1, 9]. Our focus here will be on those which are finitely generated and projective as modules. Not all our results require that the module be finitely generated, and we indicate where this assumption is not needed. We recall, however, that when the ring is connected and Noetherian, as is often the case, then Bass has shown that non-finitely generated projective modules are free [4, Cor. 4.5, p. 31]; when this happens, our relevant results are immediate.

Later in this introduction we will summarize the results to be obtained in this article. But first we would like to reflect on the conceptual framework of our investigation.

Just as a module over a commutative ring is a (concrete) representation of the ring as endomorphisms of an abelian group, a differential module over a commutative ring is both a representation of the ring as endomorphisms of an abelian group and a representation of the derivation as an inner derivation given by a designated endomorphism of the group which stabilizes the image of the ring. This stabilizing condition is strong and can be obscure: up to modification by endomorphisms which commute with the image of the ring, there can only be one. So differential modules as representations may not be as useful as modules are in the non-differential context. On the other hand, differential modules have proven to be a useful context in which to examine differential equations. As we will recall below, differential module structures on finitely generated free modules correspond to (linear, homogeneous, monic, matrix) differential equations, and complete sets of solutions of such equations correspond to module bases of constants.

CONTACT Lourdes Juan lourdes.juan@ttu.edu Department of Mathematics, Texas Tech University, Lubbock, TX, USA.
Dedicated to the 75th birthday of Lance W. Small.

In the case of a differential field, where all modules are free, this connection is covered in [10]; see also [5] and the references therein. In this work, we aim to extend these notions to projective modules.

Let R be a differential commutative ring with derivation D, and let P be a differential R module, finitely generated and projective as an R module. The quotient rule for differentiation implies that localizations of R are differential rings, and in particular that the coordinate rings of the basic open subsets of $\mathrm{Spec}(R)$ are differential. We can cover $\mathrm{Spec}(R)$ with finitely many basic open sets over each of which P is free as an R module. Thus on these open sets the differential structure is given by a differential equation which may or may not have a complete set of solutions. If they all do, and they are compatible, then P is extended from a module over the constants of R. We show that this happens if and only if P is a differential direct summand of a differential direct sum of copies of R.

Just as in the field case, one can force a differential module whose underlying R module is free of rank n to have a complete set of solutions by passing to a differential ring extension $S \supseteq R$ which is a faithfully flat finitely presented augmented R algebra (in fact as an algebra $S = R \otimes \mathbb{Z}[GL_n]$). We show that every differential projective module is a differential direct summand of a differential module whose underlying R module is free and then use this to show that every differential finitely generated projective module has a complete set of solutions in a faithfully flat finitely generated extension of the above type. That differential projective modules are such direct summands follows from our result that every projective R module carries a differential structure. Actually we prove this twice: first with a conceptual argument, and then, for the case when the projective module is given explicitly by an idempotent matrix, by a constructive formula.

The classification of differential projective modules, like the case of ordinary projective modules, is a question of K theory, which we describe formally. For the special case of rank one projective modules, this description presents the differential Picard group in terms of $\mathrm{Pic}(R)$ and a quotient of the additive group of R.

As a general reference to K theory of projective modules we cite [3].

Previous work of the authors [6] and the second author [9] have stressed the case where R is a field, or more generally a simple differential ring. Under the assumption of simplicity, every R finitely generated differential module is R projective [1, Theorem 2.2.1] (or see the exposition in [9, Theorem 5]). Here we consider when R is not necessarily a simple differential ring. Two important types of examples which we will use include the case that $R = \mathbb{C}[x_1, \ldots, x_n]_f$ is the coordinate ring of an affine open subset of complex affine space, with a suitable derivation, or that $R = \mathcal{O}(\mathbb{C})$ is the ring of entire functions on the complex plane. This latter object has some properties which may not be as familiar as the former, so we will observe them below.

2. Basics

Let R be a commutative ring with derivation D. The *ring of twisted differential polynomials* over R, denoted $R[X; D]$ is the R module $R[X]$ (ordinary R polynomials in one variable X) with associative, distributive multiplication determined by the rule $Xa = aX + D(a)$ for $a \in R$. By construction, R is a subring of $R[X; D]$, although R is not central in $R[X; D]$ unless D is trivial ($D = 0$), and hence $R[X; D]$ is not usually an R algebra.

A left module over $R[X; D]$ is called a *differential R module*. It is straightforward to verify that a differential R module M is an R module with an additive endomorphism D_M given by $D_M(m) = Xm$ which obeys the formula $D_M(am) = D(a)m + aD_M(m)$ for $a \in R$ and $m \in M$; and conversely. A homomorphism of differential R modules is called a *differential homomorphism*. A differential homomorphism $f : M \to N$ between differential R modules is seen to be an R module homomorphism that satisfies $f(D_M(m)) = D_N(f(m))$ for $m \in M$; and conversely. The category of differential R modules, being the category of left modules over the ring $R[X;D]$, is an abelian category. A projective object in this category is, by definition, a projective differential module. A *constant* of a differential ring or module is an element of derivative zero. The constants of R form a subring denoted R^D; the constants of a differential R module M form an R^D submodule of M denoted M^D. For a differential module M, there is a map $R\otimes_{R^D}M^D \to M$. If this map is surjective, we say that M is *constantly generated*. If this map is an isomorphism, we say that M is *induced* [from constants]. In Example 3, we show that constantly generated need not be induced. Suppose that M is finitely generated free as an R module and that it has an R module basis $\{x_1, \ldots, x_n\}$ consisting of constants. Then it follows that for $r_1, \ldots, r_n \in R$ that $D(\sum r_i x_i) = \sum D(r_i)x_i$ from which it is easy to conclude that $\{x_1, \ldots, x_n\}$ is an R^D basis of M^D and hence that M is induced.

We fix the above notations.

A differential module may be finitely generated as an R module. For example, this is true of R itself, using D for the endomorphism D_R. However this is not true for (nonzero) projective differential modules.

Proposition 1. *Let P be a projective left $R[X;D]$ module which is finitely generated as an R module. Then $P=0$*

Proof. Since P is R finitely generated, it is $R[X;D]$ finitely generated, which means that there is a differential surjection $R[X;D]^{(n)} \to P$. Since P is projective, this surjection differentially splits and P can be regarded as a differential submodule of $R[X;D]^{(n)}$. The projections $p_i : R[X;D]^{(n)} \to R[X;D]$ are all differential. If $P \neq 0$ then for some i $p_i(P)$ is a nonzero R finitely generated differential submodule of $R[X;D]$. Suppose $g_1, \ldots, g_k$ generate $p_i(P)$ as an R module. Then any element $g \in p_i(P)$ is of the form $\sum_{i=1}^k r_i g_i$. In particular, its degree in X is bounded. Let $f \in p_i(P)$ be a nonzero element with highest degree term $a_m X^m$ where $a_m \neq 0$. Since $X a_m X^m = a_m X^{m+1} + D(a_m)X^m$, the elements $Xf, X^2 f, X^3 f, \ldots$ all lie in $p_i(P)$ and have strictly increasing degrees. This contradicts boundedness of degrees, and hence we conclude that $P=0$. □

Proposition 1 shows that there will be no interesting projective differential modules that are finitely generated as R modules. We could go on to consider all differential modules which are finitely generated as R modules; for the reasons explained in the Introduction, the class of interest in this work is the differential R modules which are finitely generated and projective as R modules. We single out this terminology, which we have already been using, with a formal definition.

Definition 1. A differential R module which is finitely generated and projective as an R module is said to be *differential finitely generated projective*.

For example, R is a differential finitely generated projective R module. More generally $R^{(n)}$ with the endomorphism $D_0((r_1, \ldots, r_n)) = (D(r_1), \ldots, D(r_n))$ is a differential finitely generated projective R module.

If M is any R module, and D_1 and D_2 are additive endomorphisms of M which make M a differential R module, then it is elementary to check that $D_1 - D_2$ is an R module endomorphism of M. Conversely, if T is any R module endomorphism of M then the additive endomorphism $D_1 + T$ is a differential R module structure on M.

We apply this observation to differential R modules which are finitely generated free as R modules, say of rank n: let P be any differential finitely generated projective module which is free as an R module, and suppose that $\{x_1, \ldots, x_n\}$ is an R module basis of P. We use the basis to identify P with $R^{(n)}$, and use D_P for the differential structure on both. Thus $D_P = D_0 + T$ where T is an R module endomorphism. T is given by multiplication by a matrix A, so we conclude that $D_P((r_1, \ldots, r_n)) = D_0(r_1, \ldots, r_n) + (r_1, \ldots, r_n)A$. Conversely, we know that for any $n \times n$ matrix A, this formula defines a differential R module structure on $R^{(n)}$. For future reference, we will denote this differential module $P(A)$ and denote its differential structure D_A (rather than $D_{P(A)}$). Note that for $P(0)$ the notation D_0 is now unambiguous.

For use below, we now record when modules $P(A)$ and $P(B)$ are differentially isomorphic.

Occasionally it will be convenient to use another notation for D_0: $(\cdot)'$ applied to a tuple (or matrix) means to apply D to each entry. (So in the above $D_0((r_1, \ldots, r_n)) = (r_1, \ldots, r_n)'$.)

Proposition 2. *Let A and B be $n \times n$ matrices over R. Then $P(A)$ and $P(B)$ are isomorphic differential modules if and only if there is an invertible $n \times n$ matrix C over R such that $C' = AC - CB$.*

Proof. Let $P(A)$ and $P(B)$ be isomorphic as differential modules and suppose the isomorphism is given by the invertible matrix C. Then for all $x \in P(A)$ $D_B(xC) = D_A(x)C$. Expanding both sides and canceling the common term $x'C$ from both sides gives $xC' + xCB = xAC$ for all x, so that C satisfies the equation in the statement of the Proposition. Conversely, by reversing the above calculations we see that multiplication by any C satisfying the equation is a differential isomorphism. □

We will need the case $n = 1$ of Proposition 2 so we record it as a corollary.

Corollary 1. *Let a and b be elements of R. Then $P(a)$ and $P(b)$ are isomorphic differential modules if and only if there is a unit $u \in R$ such that $u'u^{-1} = a - b$*

The expression $u'u^{-1}$ that appears in Corollary 1 is called the *logarithmic derivative* of u and usually denoted $\mathrm{dlog}(u)$. It has the property that $\mathrm{dlog}(uv) = \mathrm{dlog}(u) + \mathrm{dlog}(v)$.

Now let $\underline{m} \in P(A)$. It will be a constant provided that $D_A(\underline{m}) = \underline{m}' + \underline{m}A = 0$. In other words, $\underline{m}$ is a solution of the differential equation $y' = -yA$. Here y is thought of as a $1 \times n$ matrix.

If B is any $r \times n$ matrix then $D_A(\underline{m}B) = (\underline{m}B)' + \underline{m}BA$, and $(\underline{m}B)' = \underline{m}'B + \underline{m}B'$. If we apply this where $\underline{m}$ ranges over the standard basis tuples, then the rows of B are constants provided $B' + BA$ is the zero matrix. In other words B is a solution of the matrix differential equation $Y' = -YA$; here Y is an $r \times n$ matrix. Thus $P(A)$ will have a basis of constants provided the matrix differential equation $Y' = -YA$ for Y $n \times n$ has an invertible solution. Having an invertible solution Z to the matrix differential equation $Y' = -AY$ in the classical cases where R is a differential field is known as having a complete set of solutions to the matrix differential equation.

There may not be such a matrix Z over R. However we can always adjoin elements to R to obtain such a matrix: let z_{ij}, $1 \leq i, j \leq n$ be inderminates over R and form the polynomial ring $R[z_{ij}] := R[z_{1,1}, \ldots z_{n,n}]$. Define a derivation on this polynomial ring so that if Z is the $n \times n$ matrix over it with i, j entry z_{ij} then $Z' = -ZA$. By the above, the rows of Z are constants in $R[z_{ij}] \otimes_R P(A)$. If we further make Z be invertible by localizing $R[z_{ij}]$ at the determinant $d = \det(Z)$ then the rows of Z become a basis of constants of $R[z_{ij}][d^{-1}] \otimes_R P(A)$. (This construction is the same as the first steps of the construction of the Picard–Vessiot ring extension for the module $P(A)$; see [10] and [8].) For latter reference we denote this differential ring $S(A)$. Note that, by construction, $S(A) \otimes_R P(A)$ has a basis of constants.

Note that $S(A)$, which as a ring is $R \otimes \mathbb{Z}[GL_n]$ is faithfully flat and finitely generated as an R algebra and there is an R algebra augmentation $\epsilon : S(A) \to R$ determined by $Z \mapsto I_n$ (which is not a differential augmentation, of course). And if R happens to be the coordinate ring of an affine open subset of affine space, so is $S(A)$, although the ambient affine spaces are not the same.

Recall that we have termed a differential finitely generated projective module M *induced* provided $R \otimes_{R^D} M^D \to M$ is an isomorphism and have remarked that a differential finitely generated projective R module which has a basis of constants is induced. We have the following characterization of induced modules:

Theorem 1. *Let M be a differential finitely generated projective R module. Then M is a direct summand of a finite number of copies of the differential module R if and only if there exists a finitely generated projective R^D module M_0 and a differential isomorphism $R\otimes_{R^D}M_0 \cong M$.*

Proof. Let M and N be differential modules with $M\oplus N \cong R^{(n)}$. Passing to constants, we have $M^D\oplus N^D \cong (R^D)^{(n)}$ which implies that M^D is a finitely generated projective R^D module. There are differential maps $R\otimes_{R^D}M^D \to M$ and $R\otimes_{R^D}N^D \to N$. Their direct sum is a map $(R\otimes_{R^D}M^D)\oplus(R\otimes_{R^D}N^D) \to M\oplus N$ which is the isomorphism $R\otimes_{R^D}(R^D)^{(n)} \to R^{(n)}$. It follows that both the summand maps are isomorphisms, and in particular $R\otimes_{R^D}M^D \to M$ is an isomorphism.

Conversely, if M_0 is a finitely generated projective R^D module, then there is a finitely generated projective R^D module N_0 such that $M_0\oplus N_0 \cong (R^D)^{(m)}$ for some m. Then tensoring with R over R^D shows that $(R\otimes_{R^D}M_0)\oplus(R\otimes_{R^D}N_0) \cong R\otimes_{R^D}(R^D)^{(n)} = R^{(n)}$. □

We note that the proof of Theorem 1 applies when we replace n by an arbitrary cardinal in the direct implication and similarly drop the restriction to finite generation (and replace m by whatever cardinal is necessary) in the converse.

The first half of the proof actually shows that a differential direct summand (a differential submodule of a differential module which is a direct summand and has a complement which is differential) of an induced module is induced, applied to the special case where the ambient module is a finite number of copies of the differential module R.

Theorem 1 suggests that we consider the functor $R\otimes_{R^D}(\cdot)$, which takes finitely generated projective R^D modules to differential finitely generated projective R modules: Theorem 1 says what the image is on objects. For later use, we record the following property of this functor.

Lemma 1. *Let M_0 be a finitely generated projective R^D module. Then $M_0 \to (R\otimes_{R^D}M_0)^D$ by $m\mapsto 1\otimes m$ is a bijection.*

Proof. The map $M_0 \to (R\otimes_{R^D}M_0)^D$ is natural in M_0 and additive. So, as usual with finitely generated projective modules, it suffices to prove bijection for the case $M_0 = R^D$, where it is trivial since $R\otimes_{R^D}R^D = R$. □

In the special case that all differential finitely generated projective R modules are direct sums of copies of R differentially, Theorem 1 shows that all such modules are induced.

Corollary 2. *Assume that all differential finitely generated projective R modules are direct summands of copies of R. Then all differential finitely generated projective R modules are of the form $R\otimes_{R^D}M_0$ where M_0 is a finitely generated projective R^D module.*

Theorem 1 can be used to construct examples of differential finitely generated projective modules. If T is any commutative ring, and $R = T[z, z^{-1}]$ is the ring of Laurent polynomials over T, then R is a differential ring with derivation determined by $D(z) = z$ and $D(T) = 0$. There is a ring homomorphism $R \to T$ given by $z\mapsto 1$. If T is an integral domain of characteristic 0, then $R^D = T$. Thus differential finitely generated projective R modules which are direct summands of finitely many copies of R are all of the form $R\otimes_T P$ for some finitely generated projective T module P. By varying T we can obtain examples of various types.

3. All projectives are differential

Next, we see that every projective R module is the underlying module of a differential projective module.

Theorem 2. *Let M be a projective R module. Then there is an additive endomorphism $D_M : M \to M$ that makes M a differential R module.*

Proof. Let $R[\varepsilon]$ denote the ring of dual numbers over R (The quotient $R[t]/Rt^2$ of the polynomial ring with $\varepsilon = t + Rt^2$). The derivation D of R defines a ring homomorphism $\Phi : R \to R[\varepsilon]$ by the formula $r \mapsto r + D(r)\varepsilon$. Note that Φ makes $R[\varepsilon]$ an R algebra via Φ. Consider the $R[\varepsilon]$ module $R[\varepsilon] \otimes_\Phi M$, which we denote $M[\varepsilon]$, which is an R module via Φ. The projection $\Psi_M : M[\varepsilon] \to M$ by $m + n\varepsilon \mapsto m$ satisfies $\Phi(r)(m + n\epsilon) = (r + D(r)\varepsilon)(m + n\varepsilon) = rm + (rn + D(r)m)\varepsilon \mapsto rm$ and hence is R linear. As M is a projective R module, there is an R module homomorphism $\psi_M : M \to M[\varepsilon]$ such that $\Psi_M(\psi_M(m)) = m$ for all $m \in M$. Note that if $\psi_M(m) = a + b\varepsilon$ then $m = \Psi_M(\psi_M(m)) = \Psi_M(a + b\varepsilon) = a$. We define D_M by $\psi_M(m) = m + D_M(m)\varepsilon$. Since ψ_M is an R homomorphism, $\psi_M(rm) = \Phi(r)\psi_M(m)$, so $rm + D_M(rm)\varepsilon = (r + D(r)\varepsilon)(m + D_M(m)\varepsilon)$ which implies that $D_M(rm) = D(r)m + rD_M(m)$. Similarly, since $\psi_M(m + n) = \psi_M(m) + \psi_M(n), D_M(m + n) = D_M(m) + D_M(n)$. Thus D_M is the desired additive endomorphism. $\square$

We note that if the projective module M in Theorem 2 is finitely generated then D_M makes M a differential finitely generated projective module.

As noted in the introduction, we also have a constructive version of Theorem 2 for finitely generated projective modules which are presented as the image of an idempotent matrix. Every finitely generated projective module may be so given: let M be a finitely generated projective R module, and let $p : R^{(n)} \to M$ be an R module surjection. Since M is projective, there is an R module splitting homomorphism $q : M \to R^{(n)}$ such that $pq = \mathrm{id}_M$. Let $e = qp \in \mathrm{End}_R(R^{(n)})$. Then $e^2 = e$ so e is idempotent and the image of e is isomorphic to M as an R module. We identify $\mathrm{End}_R(R^{(n)})$ with the matrix ring $M_n(R)$ so that e becomes an idempotent matrix. Moreover, $M_n(R)$ is a non–commutative differential ring via the derivation $(\cdot)'$.

We have the following lemma about idempotents in non–commutative differential rings:

Lemma 2 *Let $\mathcal{A}$ be a non–commutative differential ring with derivation $(\cdot)^\delta$ and let E be an idempotent of $\mathcal{A}$. Then $D(x) = x^\delta + [[E, E^\delta], x]$ for $x \in \mathcal{A}$ is a derivation of $\mathcal{A}$ such that $D(E) = 0$ and $D(x) = x^\delta$ if x is in the center of $\mathcal{A}$.*

Proof. The only conclusion that needs proof is that $D(E) = E^\delta + [[E, E^\delta], E] = 0$. To see this, we begin by differentiating $E = E^2$ which implies that $E^\delta = EE^\delta + E^\delta E$. Multiply this equation on both sides by E and we have $EE^\delta E = EE^\delta E + EE^\delta E$ which implies that $EE^\delta E = 0$. Expanding $[[E, E^\delta], E]$ we have $[EE^\delta - E^\delta E, E] = EE^\delta E - E^\delta E - EE^\delta + EE^\delta E = -(EE^\delta + E^\delta E) = -E^\delta$ $\square$

Lemma 2 asserts that $E^\delta + [[E, E^\delta], E] = 0$ or $E^\delta = [E, [E, E^\delta]]$. We apply this to $\mathcal{A} = M_n(R)$ with $(\cdot)^\delta = (\cdot)'$ and $E = e$. We use the matrix $A = [e, e']$ to make $R^{(n)}$ the differential module $P(A)$. Then for $x \in P(A)$ $D(xe) = (xe)' + xeA = x'e + xe' + xe[e, e']$ while $D(x)e = (x' + xA)e = x'e + x[e, e']e$. Thus $D(xe) - D(x)e = xe' + x(e[e, e'] - [e, e']e = x(e' - [e, [e, e']]) = 0$ It follows that e is an (idempotent) differential endomorphism of $P(A)$ and hence that its image (namely M) is a differential direct summand of $P(A)$ (with complementary differential summand the image of $1 - e$). We summarize:

Theorem 3. *Let M be a finitely generated projective R module and let e be an idempotent $n \times n$ matrix over R such that the image of e is isomorphic to M. Then e is an idempotent differential endomorphism of $P([e, e'])$ and hence M is a differential direct summand of $P([e, e'])$. Explicitly, for $xe \in M, D_M(xe) = (xe)' + xe[e, e']$.*

Theorem 2 implies that the additive (and multiplicative) trivialization theorems for projective modules apply to differential projective modules.

Corollary 3. *Let M be a differential finitely generated projective R module. Then there is a differential finitely generated projective R module N such that $M \oplus N$ as an R module is free of finite rank.*

Proof. There is a finitely generated projective R module N such that the R module $M \oplus N$ is free of finite rank. By Theorem 2, there is a D_N that makes N a differential module. Then $M \oplus N$ is a differential module using the differential structures of M and N. □

If in Corollary 3 we drop the requirement that M be finitely generated, the proof still applies, except without the conclusion that N or $M \oplus N$ are finitely generated.

Corollary 4. *Let P be a differential finitely generated projective R module, and assume P is faithfully projective as an R module. Then there is a differential finitely generated projective R module Q such that $P \otimes_R Q$ as an R module is free of finite rank.*

Proof. By Bass's Theorem [2, Proposition 4.6, p 476], there is a finitely generated projective R module Q such that the R module $P \otimes_R Q$ is free of finite rank. By Theorem 2, there is a D_Q that makes Q a differential module. Then $P \otimes_R Q$ is a differential module using the differential structures of P and Q. □

Corollary 3 shows that every differential finitely projective module M appears as a summand of a differential finitely generated projective module which is free as an R module, that is, of the form $P(A)$. If this latter module is constantly generated, then Theorem 1 shows that M is induced from R^D. Of course, $P(A)$ may not be constantly generated. However, we have shown that there is a differential R algebra $S(A)$ such that $S(A) \otimes_R P(A)$ is constantly generated, which implies that $S(A) \otimes_R M$ is a differential direct summand of a constantly generated module.

Thus we then have the following theorem:

Theorem 4. *Let M be a differential finitely generated projective R module. Then there is a differential R algebra S, finitely generated, faithfully flat, and augmented as an R algebra, such that $S \otimes_R M \cong S \otimes_{S^D} M_0$ where M_0 is a finitely generated projective S^D module.*

Proof. By Corollary 3 M is a differential direct summand of a differential module P which is finitely generated and free as an R module, say of rank n. If a basis is chosen for P, then there is a matrix $A \in M_n(R)$ such that $P \cong P(A)$. Thus M can be considered as a differential direct summand of $P(A)$. Let $S = S(A)$. Since $S \otimes_R P(A)$ is a direct sum of copies of S as a differential module, and $S \otimes_R M$ is a direct summand of $S \otimes_R P(A)$, by Theorem 1 $S \otimes_R M \cong S \otimes_{S^D} M_0$ for some finitely generated projective S^D module M_0. □

Because, in the notation of Theorem 4, S is faithfully flat over R, M can be recovered from $S \otimes_R M$ plus the appropriate descent data. This applies to M as a differential module as the standard descent data is differential. Once we have passed to S, then the extension of M becomes induced (tensored-up) from the constants of S. Thus differential finitely generated projective R modules are obtained from descent of induced–from–constants modules over differential extensions of R which are faithfully flat finitely generated augmented R algebras. We can further restrict S to be of the type $S(A)$, which means the extensions in question are indexed by $n \times n$ matrices over R (for all n).

4. *K*–theory

The class of differential finitely generated projective R modules is closed under (R module) direct sum. Direct sum can be used to make the set of isomorphism classes of differential finitely generated projective modules into a monoid: If $[\cdot]$ denotes differential isomorphism class then $[M] +$

$[N] := [M \oplus N]$. This is an associative and commutative operation, with identity $[0]$. Following the usual conventions, we denote the most general group to which this monoid maps $K_0^{\text{diff}}(R)$, which we call the *differential K group* of differential projective modules. We denote the image of the isomorphism class $[M]$ in $K_0^{\text{diff}}(R)$ by the same symbol, $[M]$. If S is a differential R algebra there is a homomorphism $K_0^{\text{diff}}(R) \to K_0^{\text{diff}}(S)$ induced from tensoring over R with S. Any element of $K_0^{\text{diff}}(R)$ can be written $[M]-[N]$, where M and N are differential finitely generated projective modules. By Corollary 3, we can choose a P such that $N \oplus P$ is free as an R module. Since $[M]-[N] = [M \oplus P]-[N \oplus P]$ we can always assume that N is R free, so that any element of $K_0^{\text{diff}}(R)$ is of the form $[M]-[P(A)]$ for some matrix A. If P and Q are differential finitely generated projective modules then $[P] = [Q]$ in $K_0^{\text{diff}}(R)$ if and only if there is a differential finitely generated projective module M such that $P \oplus M \cong Q \oplus M$. By adding an appropriate differential module to M we can, by Corollary 3 again, assume that M is R free,

The group $K_0^{\text{diff}}(R)$ is related to the usual K-theory of R and R^D. Regarding the former, there is a homomorphism $K_0^{\text{diff}}(R) \to K_0(R)$ which, by Theorem 2, is surjective. An element $[M]-[P(A)]$ of $K_0^{\text{diff}}(R)$ lies in the kernel provided that, as R modules, M and $P(A)$ are stably isomorphic, which means that M is stably free, say $M \oplus R^{(m)} \cong R^{(n)}$. In $K_0^{\text{diff}}(R), [M]-[P(A)] = [M \oplus R^{(m)}]-[P(A) \oplus R^{(m)}]$. Thus every element of the kernel has the form $[P(C)]-[P(D)]$ where C and D are matrices of the same size. Conversely, every such matrix pair yields an element of the kernel. If C and D are matrices of the same size over R and both $P(C)$ and $P(D)$ have bases of constants then they are isomorphic and $[P(C)]-[P(D)] = 0$ in $K_0^{\text{diff}}(R)$. If we extend scalars to $S = S(C) \otimes_R S(D)$ then the extensions of $P(C)$ and $P(D)$ will have bases of constants. If we do this for all pairs of matrices, then for all pairs $[P(C)]-[P(D)] = 0$ and hence the kernel of the homomorphism from the differential K group to the ordinary K group is trivial.

Explicitly:

Proposition 3. *Let $S_1(R)$ be the infinite tensor product of the algebras $S(A)$ as A ranges over all the matrices, of all sizes, with entries in R. $S_1(R)$ is a faithfully flat augmented R algebra, and the image of*

$$K_0^{\text{diff}}(R) \to K_0^{\text{diff}}(S_1(R))$$

is isomorphic to $K_0(R)$.

Proof. If C and D are matrices of the same size over R then, as we have noted, $[P(C)]-[P(D)]$ has image zero under $K_0^{\text{diff}}(R) \to K_0^{\text{diff}}(S_1(R))$. Thus the kernel of $K_0^{\text{diff}}(R) \to K_0(R)$ is contained in that of $K_0^{\text{diff}}(R) \to K_0^{\text{diff}}(S_1(R)$. It is clear that $S_1(R)$ is augmented and faithfully flat. The augmentation induces a homomorphism $K_0(S_1(R)) \to K_0(R)$. If we precede this by the maps $K_0^{\text{diff}}(R) \to K_0^{\text{diff}}(S_1(R))$ and $K_0^{\text{diff}}(S_1(R)) \to K_0(S_1(R))$ the composite is $K_0^{\text{diff}}(R) \to K_0(R)$ which shows that the kernels coincide, completing the proof. □

We can say that Proposition 3 says that, after extension from R to $S_1(R)$, the differential and ordinary K theory of R are the same. We can iterate this process:

Corollary 5. *Let $S_{i+1}(R) = S_1(S_i(R)), i = 1, 2, 3, \ldots$. Let $S_\infty(R) = \lim S_i(R)$. Then $S_\infty(R)$ is a faithfully flat R algebra and*

$$K_0^{\text{diff}}(S_\infty(R)) \to K_0(S_\infty(R))$$

is an isomorphism.

Corollary 5 is proven by taking the direct limit of the maps in Proposition 3.

Regarding R^D, there is a homomorphism $K_0(R^D) \to K_0^{\text{diff}}(R)$ given by tensoring over R^D with R. By Theorem 1 the image of this homomorphism is given by (differences of) objects which occur as summands of $m[R]$ for $m = 1, 2, 3, \ldots$. Moreover, there may be a kernel, ultimately due to the fact that a surjective differential homomorphism need not be surjective on constants.

Applying Theorem 4 then yields the following:

Corollary 6. *Let $x \in K_0^{\mathrm{diff}}(R)$. Then there is a differential R algebra S, faithfully flat, finitely generated, and augmented as an R algebra such that under $K_0^{\mathrm{diff}}(R) \to K_0^{\mathrm{diff}}(S)$ the image of x lies in the image of $K_0(S^D) \to K_0^{\mathrm{diff}}(S)$.*

Proof. Let $x = [M_1]-[M_2]$ where M_i is a differential finitely generated projective module. Let S_i be the algebra provided for M_i by Theorem 4, and let $S = S_1 \otimes S_2$. Then both M_1 and M_2 are differential direct summands of direct sums of copies of S and hence induced from S^D by Theorem 1. □

If we apply the proof of Corollary 6 to the case where R is replaced by $S_\infty(R)$ we see that $K_0(S_\infty(R)^D) \to K_0^{\mathrm{diff}}(S_\infty(R))$ is surjective.

The situation for differential projective modules which are rank one as R modules is somewhat simpler. Differential isomorphism classes of such modules form a multiplicative monoid under the relation $[I][J] := [I \otimes J]$ with $[R]$ acting as an identity, If I is a differential rank one projective module so is $\mathrm{End}_R(I)$. Consider the endomorphism T_a of I given by multiplication by $a \in R$: by definition, $D(T_a)(x) = D(T_a(x))-T_a(D(x)) = D(ax)-aD(x) = D(a)x$. Thus $D(T_a) = T_{D(a)}$, which means that $R \to \mathrm{End}_R(I)$ is a differential map. Since it is an R isomorphism, this shows that $[\mathrm{End}_R(I)] = [R]$. The usual isomorphism $I^* \otimes I \to \mathrm{End}_R(I)$ is differential, showing that $[I^*][I] = [R]$. This means the isomorphism classes of differential rank one projective R modules form a group, which we denote $\mathrm{Pic}_{\mathrm{diff}}(R)$. For the special case $I = P(a)$ and $J = P(b)$ for $a, b \in R$, we have $I \otimes J \cong P(a+b)$, so that $a \mapsto [P(a)]$ is a group homomorphism from the additive group of R, which we denote R^+, to $\mathrm{Pic}_{\mathrm{diff}}(R)$.

There is a homomorphism $\mathrm{Pic}_{\mathrm{diff}}(R) \to \mathrm{Pic}(R)$ sending $[I]$ to the isomorphism class of I. This is surjective by Theorem 2. The kernel, which we denote $\mathrm{Pic}^0_{\mathrm{diff}}(R)$, consists of I's which are free as R modules, namely those of the form $P(a)$ for $a \in R$, in other words R with derivation $D_a(r) = r' + ar$. The image of $R^+ \to \mathrm{Pic}_{\mathrm{diff}}(R)$ is thus $\mathrm{Pic}^0_{\mathrm{diff}}(R)$. Thus $\mathrm{Pic}_{\mathrm{diff}}(R)$ is an extension of $\mathrm{Pic}(R)$ by the image of R^+, as the following theorem records.

Theorem 5. *There is an exact sequence*

$$0 \to R^+/\mathrm{dlog}(R^\times) \cong \mathrm{Pic}^0_{\mathrm{diff}}(R) \to \mathrm{Pic}_{\mathrm{diff}}(R) \to \mathrm{Pic}(R) \to 1$$

Proof. The only assertion that needs proof is the isomorphism $R^+/\mathrm{dlog}(R^\times) \cong \mathrm{Pic}^0_{\mathrm{diff}}(R)$. We already know that $R^+ \to \mathrm{Pic}^0_{\mathrm{diff}}(R)$ by $a \mapsto [P(a)]$ is surjective. By Corollary 1. $[P(a)] = [P(b)]$ are isomorphic differential modules if and only if there is a unit $u \in R$ such that $a-b = \mathrm{dlog}(u)$, which shows that the kernel of the surjection is $\mathrm{dlog}(R^\times)$. □

5. Examples

This section collects some examples illustrating some extreme cases of differential rings. We intend to pursue investigations of K-theory calculations in future work.

Example 1. *Let $R = \mathbb{C}$ with $D = 0$.*

In this case all finitely generated R modules are projective, indeed free, and for a differential R module M D_M is R linear. Thus differential finitely generated projective R modules are finite dimensional vector spaces with designated endomorphisms. The structure, classification, and K-theory of these modules is thus the same as that of complex matrices under conjugation.

Example 2. Let $R = \mathcal{O}(\mathbb{C})$ (the ring of entire functions on the complex plane in the variable z) with $D = \frac{d}{dz}$.

It is a consequence of Weierstrass Theory (see [7, Chapter 1]) that finitely generated ideals of R are principal. Suppose f is an entire function and that the principal ideal Rf is a differential ideal. Suppose that $f(\alpha) = 0$ for some $\alpha \in \mathbb{C}$, and suppose the order of the zero of f at α is $n > 0$. Then f' has a zero at α of order $n - 1$. But since Rf is a differential ideal, f is a factor of f', so that f' has a zero at α of order at least n. We conclude that f has no zeros, and hence is a unit, so that $Rf = R$ is the only finitely generated differential ideal of R.

On the other hand, R has proper differential ideals: again by Weierstrass Theory, there is an entire function g which has a zero at n of order n for every natural number $n \in \mathbb{N}$. This means that for any m the functions $g^{(i)}, i = 0, 1, \ldots, m$ have infinitely many common zeros at $m+1, m+2, \ldots$ and so the ideal of R generated by $g, g', \ldots, g^{(m)}$ does not contain 1. It follows that the differential ideal I of R generated by g is proper. We note that the differential R module R/I is finitely generated as an R module, but not projective as an R module, since R is an integral domain, and hence its only idempotents are 0 and 1, so the surjection $R \to R/I$ can't split. This is thus an example of a non-projective differential finitely generated module over a differential ring that has no non-trivial finitely generated differential ideals.

Because finitely generated ideals of R are principal, finitely generated projective R modules are free. This means that the differential R modules which are finitely generated and projective as R modules are R free, so have the form $P(A)$ for a suitable matrix A over R. [Monic, homogeneous] linear differential equations with entire coefficients have an entire solution, which implies that $P(A)$ has a basis of constants, and hence is a differential direct sum of copies of R.

By Corollary 2, this means that every differential finitely generated projective R module is induced from $R^D = \mathbb{C}$. (Note: this is $\mathbb{C}$ as a ring, not as a differential ring as in Example 1.)

Example 3. Let R be the localized polynomial ring $\mathbb{C}[a,b,c][h^{-1}]$ where $h = a^2b^2 + a^2c + b^2c$ and $D(a) = a$, $D(b) = b$, $D(c) = c$, and $D(\alpha) = 0$ for $\alpha \in \mathbb{C}$.

Finitely generated projective R modules are free, so all of the form $P(A)$ for some A. Let A be the 2×2 complex matrix $-I_2$, and consider the differential R module $M = P(A)$, which is free of rank 2 as an R module. We will see that M is constantly generated but not induced.

For $\underline{x} \in M, D_M(\underline{x}) = \underline{x}' + \underline{x}A$. Then $D_M((a,0)) = (0,0), D_M((0,b)) = (0,0)$, and $D_M((c,c)) = (0,0)$, so $(a,0), (0,b)$, and (c, c) are constants. Moreover, $a(a,0) + (c,c) = (a^2+c, c)$ and $b(0,b) + (c,c) = (c, b^2+c)$ are the rows of a 2×2 matrix whose determinant h is a unit of R, and hence they form a basis of M. Thus the constants $(a,0), (0,b)$, and (c, c) span M.

In other words, $T : R^{(3)} \to M$ by $(x,y,x) \mapsto x(a,0) + y(0,b) + z(c,c) = (xa + zc, yb + zc)$ is a differential surjection. The matrix of T is

$$\begin{bmatrix} a & 0 & c \\ 0 & b & c \end{bmatrix}.$$

It is easy to calculate a right R module inverse to T:

$$\begin{bmatrix} a & 0 & c \\ 0 & b & c \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a^2+c & c \\ c & b^2+c \end{bmatrix} = C$$

and $\det(C) = h$ (unit of R). If T had a differential right inverse, then M would be a differential direct summand of $R^{(3)}$ and hence, by Theorem 1, induced. We see that this is not the case.

Suppose $(x,y) \in M$ is a constant. Then $(0,0) = D_M((x,y)) = (x' - x, y' - y)$ so $D(x) = x$ and $D(y) = y$. Now suppose (x, y) and (z, w) are an R basis of M consisting of constants. The determinant of the matrix with rows (x, y) and (z, w) is $g = xw - yz$ and $D(g) = 2g$. Moreover, g is a unit. Now R is the localization of a unique factorization domain at an irreducible polynomial, so its only units are complex multiples of powers of h. So suppose $g = \alpha h^k, \alpha \in \mathbb{C}$, in the polynomial ring. Then $2\alpha h^k = 2g = D(g) = \alpha k h^{k-1} D(h)$. Therefore $D(h) = 2k^{-1}h$ is a constant multiple of h.

Moreover $D(g) = 2g$ implies that $g \notin R^D$, since g is a unit, and therefore $k \neq 0$. But $D(h) = 4a^2b^2 + 3b^2c + 3a^2c$ is not a polynomial multiple of h. Thus we conclude that M does not have a basis of constants, despite the fact that it is constantly generated. We now show that this implies M is not induced by showing that $R^D = \mathbb{C}$. Consider an element of R with derivative zero. We can write this as a fraction $\frac{p}{q}$ where p and q are relatively prime polynomials and q is a power of h. Since $\frac{p}{q}$ is a constant, $qD(p) - pD(q) = 0$. If $q \neq 1$ this implies that q is a factor of $D(q)$ in the polynomial ring $\mathbb{C}[a, b, c]$. Since $q = h^k$ for some $k > 0$, we have h^k as a factor of $kh^{k-1}D(h)$ in $\mathbb{C}[a, b, c]$ so that h is a factor of $D(h)$ in $\mathbb{C}[a, b, c]$; as noted, this is not possible. Thus $q = 1$ and $D(p) = 0$. Write $p = \sum_{i=0}^n p_i$, where p_i is the term in p of total degree i. Then $D(p) = \sum^n ip_i$ so $D(p) = 0$ implies that $p = p_0 \in \mathbb{C}$.

Example 4. Let F be a differential field whose field of constants C is algebraically closed of characteristic zero and let $E \supset F$ be a Picard–Vessiot extension with differential Galois group $G = PGL_n(C)$, n > 1. Let R be the Picard–Vessiot ring of the extension, and assume that as a ring $R = F \otimes_C C[PGL_n]$. (Such examples are known and easy to construct since we have no constraint on F).

Since $\mathrm{Pic}(C[PGL_n]) = \mathbb{Z}/n\mathbb{Z}$ we conclude that $\mathrm{Pic}(R)$ contains $\mathbb{Z}/n\mathbb{Z}$ and in particular is non-–zero. Let $I \subset C[PGL_n]$ be a (projective) ideal generating $\mathrm{Pic}(C[PGL_n])$ and let $J = F \otimes I$ be the corresponding rank one projective ideal of R. By Theorem 2 J carries a differential structure, although not as a differential ideal of R (R is differentially simple). Moreover, since R is differentially simple, the cyclic R submodule generated by a constant in any differential module is a differential direct summand. Thus J, as a differential module, can't contain any constants.

The units of $C[PGL_n]$ are elements of C (in general the units of $C[G]$ are constants times characters of G) from which it follows that the units of R are non-zero elements of F, so that $\mathrm{dlog}(R^\times) = \mathrm{dlog}(F^\times)$. Thus $R/\mathrm{dlog}(R^\times)$ contains the countable dimensional F vector space spanned by the augmentation ideal of R, and hence $\mathrm{Pic}^0_{\mathrm{diff}}(R)$ is highly infinite.

ORCID

Lourdes Juan http://orcid.org/0000-0002-4208-7330

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